On the statistical description of classical open systems with integer variables by the Lindblad equation

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We propose the consistent statistical approach to consider a wide class of classical open systems whose states are specified by a set of positive integers(occupation numbers).Such systems are often encountered in physics, chemistry, ecology, economics and other sciences.Our statistical method based on ideas of quantum theory of open systems takes into account both discreteness of the system variables and their time fluctuations - two effects which are ignored in usual mean field dynamical approach.The method let one to calculate the distribution function and (or)all moments of the system of interest at any instant.As descriptive examples illustrating the effectiveness of the method we consider some simple models:one relating to nonlinear mechanics,and others taken from population biology .In all this examples the results obtained by the method for large occupation numbers coincide with results of purely dynamical approach but for small numbers interesting differences and new effects arise.The possible observable effects connected with discreteness and fluctuations in such systems are discussed.

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I. INTRODUCTION

Among the vast number of classical open dynamical systems under consideration in physics, chemistry, biology, economics and other sciences there are many such whose states in accordance with the sense of the problem are specified by a set of integer variables $(\{n_i\})$, where i = 0,1,2,..N (N-number of degrees of freedom). For example in physics n_i- are occupation numbers of cell states in phase space, in chemistry - numbers of molecules of reactive elements, in ecology -numbers of individuals in populations which live in the area and interact with other populations, in economics the number of companies operating on the market. Since all these systems are classical their dynamics as a rule is described by a system of differential equations of the form

$$\frac{dn_i}{dt} = F_i\left(\{n_\alpha\}\right), \tag{1}$$

where $F_i(\{n_\alpha\})$ -some nonlinear functions, depending on concrete problem. Obviously notation Eq. (1) implies that variables n_i in system Eq. (1) -are considered as continuous. In the case when all $n_i \gg 1$ such approach can be easy justified. From the physical point of view the system of equations (1) corresponds to mean field approximation and n_i are occupation numbers averaged over some appropriate statistical ensemble. In the case when all $n_i \gtrsim 1$, dynamical description becomes inadequate and the question naturally arises: is there consistent statistical approach which takes into account both discreteness of variables n_i and their time fluctuations that may be not small. In addition it is naturally to demand that such approach gave the same results as dynamical description in the large n_i limit. In this paper we propose such approach based on the ideas of quantum theory of open systems (QTOS) and consider some examples that demonstrate its effectiveness. The rest of the article organized as follows. In the Sect.2 we briefly describe minimal information from (QTOS) which is necessary for understanding of the method used and present main steps of our method. In Sect.3 we consider simple model of nonlinear autonomous oscillator with soft exciting mode and give its statistical description on the basis of the method proposed. All the main features of the method clearly come to light already in this representative example. In Sect.4 with the help of our approach we consider some problems from population dynamics relating to evolution of two interaction populations living in a certain area..We show that in the case of small populations statistical description leads to a number of differences from the ordinary dynamical picture. On the other hand in the case of large occupation numbers both descriptions are virtually identical. In conclusion we discuss some generalizations of the method and its possible experimental verification.

II. DESCRIPTION OF THE METHOD

In this section we briefly remind the main points of the method proposed by author earlier [1] which allows one to make the transition from known dynamical equations of classical open system to the master equation for its quantum analogue. The method based on the correspondence that can be set between quantum master equation in the Lindblad form and the Liouville equation for distribution function in phase space of the classical system of interest. This correspondence allows one using classical

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equations of motion to restore the form of all operators involved in the Lindblad equation. Thereby we can apply the procedure of quantization at least in semiclassical approximation in the case of a large class of nonhamiltonian dynamical systems. In short (all details see in [1]) the recipe of quantization proposed consists of three consecutive steps.

Step1: The input classical dynamical equations should be presented in the form allowed the quantization (FAQ). For the purposes of present paper the most convenient form is complex representation of equations of motion:

$$\frac{dz_i}{dt} = -i \cdot \frac{dH}{dz_i^*} + \sum_\alpha \left(\overline{R_\alpha} \frac{dR_\alpha}{dz_i^*} - R_\alpha \frac{d\overline{R_\alpha}}{dz_i^*} \right), \quad (2)$$

where $z_i = \frac{x_i + iy_i}{\sqrt{2}}$, $z_i^* = \frac{x_i - iy_i}{\sqrt{2}}$ are complex dynamical coordinates of the system of interest, H, R_α, $\overline{R_\alpha}$ are functions of z_i, z_i^* (H is real function, and R_α, $\overline{R_\alpha}$ are complex, $\overline{R_\alpha}$ means function which conjugate to R_α). It is necessary to emphasize that it is the most delicate step of the method because it is difficult exactly to formalize this point.

Step2. Having in hands representation Eq. (2) we can use classical function H, R_α, $\overline{R}_\alpha$ and with their help determine their quantum analogues - operators $\widehat{H}$, $\widehat{R_\alpha}$, $\widehat{R^+}$. For this purpose the variables z_i, z_i^* must be replaced by the correspondence Bose operators $\widehat{a}_i$ and $\widehat{a}_i^+$ with usual commutation rules: $\left[\widehat{a}_i, \widehat{a}_j^+\right] = \delta_{ij}$, $\left[\widehat{a}_i, \widehat{a}_j\right] = \left[\widehat{a}_i^+, \widehat{a}_j^+\right] = 0$.

Step3. The operators $\widehat{H}$, $\widehat{R_\alpha}$, $\widehat{R_\alpha^+}$ found in this manner should be substituted into the quantum Lindblad equation for the evolution of density matrix of the system:

$$\frac{d\widehat{\rho}}{dt} = -i\left[\widehat{H}, \widehat{\rho}\right] + \sum_\alpha \left[\widehat{R_\alpha} \cdot \widehat{\rho}, \widehat{R_\alpha^+}\right] + \left[\widehat{R_\alpha}, \widehat{\rho} \cdot \widehat{R_\alpha^+}\right]. \quad (3)$$

The correspondence principle guarantees us that such approach will give correct description of evolution of quantum open system at least with the accuracy up to the first order in $\hbar$.

III. REPRESENTATIVE MODEL. AUTONOMOUS NONLINEAR OSCILLATOR WITH SELF EXCITING MODE

It is convenient to demonstrate on concrete example all the features of the approach proposed. We consider an oscillator with nonlinear damping in situation when its equilibrium point loses stability and small fluctuations switch the system to the new stationary state that corresponds to the closed trajectory (limit cycle). The following system of equations gives the correct mathematical description of the behavior of the oscillator in the vicinity of bifurcation point (see [2]):

$$\begin{aligned} \frac{dx}{dt} &= \omega y + \mu x - x\left(x^2 + y^2\right), \quad (4) \\ \frac{dy}{dt} &= -\omega x + \mu y - y\left(x^2 + y^2\right), \end{aligned}$$

where x, y are coordinates of oscillator in phase space and ω -its frequency. The system of the equations of motion Eq. (4) can be written in the complex form as one equation

$$\frac{dz}{dt} = -i\omega z + \mu z - 2z\left|z\right|^2, \quad (5)$$

where $z = \frac{x+iy}{\sqrt{2}}$. It is easy to show that equation Eq. (5) can be represented in the FAC. For this purpose we introduce the functions $H = \omega z^* z$, $R_1 = \sqrt{\mu} z^*$ and $R_2 = z^2$. One can verify by direct checking that r.h.s. of Eq. (5) can be rewritten in the form

$$\frac{dz}{dt} = -i\frac{\partial H}{\partial z^*} + \left(\overline{R_1}\frac{\partial R_1}{\partial z^*} - R_1\frac{\partial \overline{R_1}}{\partial z^*}\right) + \left(\overline{R_2}\frac{\partial R_2}{\partial z^*} - R_2\frac{\partial \overline{R_2}}{\partial z^*}\right). \quad (6)$$

According to the recipe of quantization the Lindblad equation for quantum analog of the system Eq. (4) can be written in the form

$$\frac{d\rho}{dt} = -i\left[H, \rho\right] + \left[R_1\rho, R_1^+\right] + \left[R_2\rho, R_2^+\right] + h.c, \quad (7)$$

where $\widehat{H} = \omega\widehat{a}^+\widehat{a}$, $\widehat{R}_1 = \sqrt{\mu}\widehat{a}^+$ and $\widehat{R}_2 = \widehat{a}^2$.

We are interesting in the stationary solution of Eq. 7 and from physical considerations we will seek it in the form: $\widehat{\rho} = \widehat{\rho}(\widehat{N}) = \sum |n\rangle\, \rho_n \,\langle n|$, where $|n\rangle$ are the eigenfunctions of the operator $\widehat{N} = \widehat{a}^+\widehat{a}$. It is convenient to introduce the generating function $G(u)$ for the coefficients ρ_n. By definition $G(u) = \sum_{n=0}^{n=\infty} \rho_n \cdot u^n$.One can obtain from Eq. 7 in stationary case the following equation for the function $G(u)$:

$$(1+u)\frac{d^2G}{du^2} - \mu u\frac{dG}{du} - \mu G(u) = 0. \quad (8)$$

The solution of Eq. 8 that satisfies all conditions of the problem may be represented as

$$G(u) = \frac{\Phi(1, \mu, \mu(1+u))}{\Phi(1, \mu, 2\mu)}, \quad (9)$$

where $\Phi(a, c, x)$ is well known confluent hypergeometric function (see [3]) - $\Phi(a, c, x) = 1 + \frac{ax}{c} + \frac{a(a+1)}{2!c(c+1)}x^2 + \ldots$. Note that condition $G(u) = 1$ corresponds to normalization of $\widehat{\rho_{st}}(n)-$ namely $\sum_n \rho_n = 1$. Having in hands the expression Eq. 9 for the generation function one can find the average values for any physical quantity of interest in the stationary state that is all moments of the distribution $\rho(n)$. For example moments of first and second

order are determined by relations: $\overline{n} = \sum_n n \cdot \rho_n = \frac{dG}{du}$, $\overline{n^2} - \overline{n} = \frac{d^2G}{du^2}$. (all derivatives are taken at point $u = 1$). Let us consider now the behavior of our system in two limiting cases: $\mu \gg 1$ and $\mu \ll 1$. When $\mu \gg 1$ using the asymptotic formula for $\Phi(a, c, x)$ (see [3])we obtain for $G(u)$ the next expression

$$G(u) \approx e^{\mu(u-1)} \cdot \left(\frac{1+u}{2}\right)^{1-\mu}, \tag{10}$$

The average number of quanta in stationary state $\overline{n}$ and their dispersion $\sigma = \overline{n^2} - \overline{n}^2$ in this case are: $\overline{n} = \frac{\mu+1}{2}$ and $\sigma = \frac{3\mu+1}{4}$. The relative fluctuation of quanta generated in stationary state is $\frac{\sqrt{\sigma}}{\overline{n}} = \frac{\sqrt{3\mu+1}}{\mu+1}$ tends to zero when $\mu \gg 1$. The obtained result testifies validity of deterministic approach in this case since in classical case $n_{cl} = |z|^2 = \frac{\mu}{2}$ when the system is moving along the limit cycle. Now let us consider the opposite case when $\mu \ll 1$. Using the series expansion for $\Phi(a, c, x)$ [3] we obtained for $G(u)$ the next expression (up to the first order in μ)

$$G(u) \backsimeq \frac{(2+u)(1+\mu) + \mu(1+u)^2}{3+7\mu}. \tag{11}$$

The relation Eq. 11 implies that $G(u)$ tends to $G_0(u) = \frac{2+u}{3}$ when μ tends to zero. The first two moments of the distribution $\rho(n)$ are: $\overline{n} = \frac{1}{3}$ and $\sigma = \frac{2}{9}$. The relative fluctuation of $n = \frac{\sqrt{\sigma}}{\overline{n}}$ is equal to $\sqrt{2}$ in this case. We see that in contrast to classical situation $\overline{n} \neq 0$ when μ tends to zero moreover fluctuations of occupation number turn out to be large and very essential.

It is necessary to note that in classical system Eq. 4 in addition to variable $|z|^2$ we have phase variable φ that satisfies to equation $\frac{d\varphi}{dt} = \omega$. However phase dynamics determines only velocity along limit cycle but does not influence on stationary state itself. In principle we can turn ω to zero (which implies that $H = 0$) and all foregoing results do not change. It is the point which explains why the Lindblad equation can be successfully applied for statistical description of classical open systems with integer variables. The reason is that in the case when $H = 0$ in Eq. 4, $\hbar$ may be eliminated from the Lindblad equation and wave properties of quantum system turn out to be irrelevant. After this crucial remark we can apply our approach to different classical systems with integer variables in particular population dynamics models.

IV. STATISTICAL DESCRIPTION OF POPULATION DYNAMICS MODELS.

We begin our consideration with well known Lotka-Volterra model (LVM) (see [4]) describing "the interaction" between two populations: prays and predators (for example hares and lynxes) living in the same territory. Let $n_1(t)$ and $n_2(t)$ the current numbers of preys and predators correspondently. Then input dynamical equations of the LVM can be written as

$$\begin{aligned} \frac{dn_1}{dt} &= n_1 (a - 2n_2), \\ \frac{dn_2}{dt} &= -n_2 (b - 2n_1), \end{aligned} \tag{12}$$

where coefficients a, b are positive and have clear ecological meaning. Note that for convenience we have chosen time unit thus to put coefficient of the term $n_1 n_2$ in Eq. 12 equal to 2. Now let us demonstrate that system Eq. 12 can be represented in FAQ. For this purpose we introduce two auxiliary complex variables z_1 and z_2 so that $n_1 = |z_1|^2$and $n_2 = |z_2|^2$. Let us assume that evolution z_1, z_2 in time is governed by the following system of equations:

$$\begin{aligned} \frac{dz_1}{dt} &= \lambda_1^2 z_1 - z_1 |z_2|^2, \\ \frac{dz_2}{dt} &= -\lambda_2^2 z_2 + z_2 |z_1|^2. \end{aligned} \tag{13}$$

It is easy to see that system Eq. 13 implies the following equations for n_1, n_2

$$\begin{aligned} \frac{dn_1}{dt} &= 2\lambda_1^2 n_1 - 2n_1 n_2, \\ \frac{dn_2}{dt} &= -2\lambda_2^2 n_2 + 2n_1 n_2, \end{aligned} \tag{14}$$

which is exactly coincides with Eq. 12 if one lets $a = 2\lambda_1^2$, $b = 2\lambda_2^2$.

Now if we introduce three functions $R_1 = \lambda_1 z_1^*$, $R_2 = \lambda_2 z_2$, and $R_3 = z_1 z_2^*$ it is easy to verify that Eq. 13 can be represented in the form

$$\begin{aligned} \frac{dz_1}{dt} &= \sum_{i=1}^{3} \left(\overline{R_i} \frac{\partial R_i}{\partial z_1^*} - R_i \frac{\partial \overline{R_i}}{\partial z_1^*} \right), \\ \frac{dz_2}{dt} &= \sum_{i=1}^{3} \left(\overline{R_i} \frac{\partial R_i}{\partial z_2^*} - R_i \frac{\partial \overline{R_i}}{\partial z_2^*} \right). \end{aligned} \tag{15}$$

Using the foregoing recipe of quantization we maintain that description of LVM which takes into account the discreteness of variables n_1 and n_2 and their time fluctuations is given by the next master equation

$$\frac{d\rho}{dt} = \sum_{i=1}^{3} \left[\widehat{R}_i \rho, \widehat{R}_i^+ \right] + h.c., \tag{16}$$

where $\widehat{R}_1 = \lambda_1 \widehat{a}_1^+$, $\widehat{R}_2 = \lambda_2 \widehat{a}_2$ and $\widehat{R}_3 = \widehat{a}_1 \widehat{a}_2^+$. Again we will interesting in only the solutions of Eq. 16 which have the form $\widehat{\rho} = \sum_{n_1,n_2} |n_1 n_2\rangle \rho_{n_1,n_2} \langle n_1 n_2|$. In this case for the coefficients of the expansion $\rho_{n_1 n_2}$ we obtain the next general equation

$$\frac{d\rho_{n_1n_2}}{dt} = 2\lambda_1^2\left[n_1\rho_{n_1-1,n_2} - (n_1+1)\,\rho_{n_1n_2}\right] + 2\lambda_2^2\left[(n_2+1)\,\rho_{n_1,n_2+1} - n_2\rho_{n_1n_2}\right] + 2\left[(n_1+1)\,n_2\rho_{n_1+1,n_2-1} - (n_2+1)\,n_1\rho_{n_1n_2}\right] \tag{17}$$

. It is convenient to introduce the generating function $G(u,v,t) = \sum\limits_{n_1n_2} \rho_{n_1n_2} \cdot u^{n_1} \cdot v^{n_2}$.Then after the simple algebra we find that Eq. 17 implies the next equation for $G(u,v,t)$:

$$\frac{\partial G}{\partial t} = 2\lambda_1^2(u-1)\frac{\partial}{\partial u}(uG) + 2\lambda_2^2(1-v)\frac{\partial G}{\partial v} + 2(v-u)\frac{\partial^2}{\partial u\partial v}(vG). \tag{18}$$

The Eq. 17 and Eq. 18 in principle give us all the necessary information about statistical behaviour of LVM. Now let us consider concrete results following from these equations. Note that even in stationary case it is difficult to find exact analytical solutions of Eq. 18 for arbitrary $\lambda_1$and λ_2. But in the special case when $\lambda_2^2 = 1+\lambda_1^2$ such solution easily can be found and have the form $G(u,v) = \frac{(1-\varkappa)^2}{(1-\varkappa u)(1-\varkappa v)}$ where $\varkappa = \frac{\lambda_1^2}{1+\lambda_1^2}$. Since $G(u,v) = g(u)\cdot g(v)$ it is clear that $n_1$and n_2 are independent variables in this case and $\overline{n_1} = \overline{n_2} = \frac{\varkappa}{1-\varkappa} = \lambda_1^2$.The dispersion σ_1^2 in this case is equal to $\lambda_1^2 + \lambda_1^4$ and if we calculate relative fluctuation of n_1 in stationary state we get the result: $\delta n_1 = \frac{\sqrt{\sigma_1}}{\overline{n_1}} = \sqrt{1+\frac{1}{\lambda_1^2}}$. Thereby we see this quantity is not small and can be easily measured. More detailed analysis of statistical behaviour of LVM following from Eq. 17 for arbitrary λ_1 and λ_2 will be carried out elsewhere. Here we show only that using Eq. 18 one can easy obtain a collection of explicit relations connecting different moments of distribution $\rho(n_1,n_2)$. For example if we differentiate stationary Eq. 18 with respect to u and after that put $u=v=1$ we obtain the simple relation between moments of first and second order which reads as:

$$\lambda_1^2\,(1+\overline{n_1}) - n_1 - \overline{n_1n_2} = 0. \tag{19}$$

In a similar manner by differentiating of Eq. 18 with respect to v we obtain the second relation:

$$-\lambda_2^2\overline{n_2} + \overline{n_1} + \overline{n_1n_2} = 0 \tag{20}$$

Relations Eq. 19 and Eq. 20 imply the helpful equation connecting $\overline{n_1}$ and $\overline{n_2}$ namely $\frac{\overline{n_2}}{1+\overline{n_1}} = \frac{\lambda_1^2}{\lambda_2^2}$. It is worth to note that in classical LVM Eq. 14 similar relation exists: $\frac{\overline{n_2}}{\overline{n_1}} = \frac{\lambda_1^2}{\lambda_2^2}$, so we conclude that when $\overline{n_1} \gg 1$, $\overline{n_2} \gg 1$ the results of statistical description completely coincide with pure dynamical consideration. But in the case of small numbers n_1, n_2 the difference between them may be essential. To demonstrate this distinction and also to compare the approach proposed in the present paper with usual Markovian description of such systems proposed in well known article of Nicolis and Prigogine [5] and expanded in their later book [6] it is appropriate to consider the truncated case of LVM, with $\lambda_1 = \lambda_2 = 0$. It is obvious that total number of individuals N= n_1+n_2 in this model will be constant .This fact greatly simplifies finding and analysis of solutions Eq. 18 which in this case takes the form:

$$\frac{\partial G}{\partial t} = (v-u)\frac{\partial^2}{\partial u\partial v}(vG) \tag{21}$$

It is easy to see that Eq. 21 has solutions for any integer N in the form of homogeneous polynomial in u and v of degree N , namely $G_N\,(u,v,t) = \sum\limits_k A_k\,(t)\,u^kv^{N-k}$ where coefficients A_k satisfy the normalization condition $\sum\limits_k A_k = 1$. Thereby Eq. 21 is reduced to the linear system of equations of N+1order for coefficients A_k of the form $\frac{dA_k}{dt} = L_{km}A_m$, where matrix elements L_{km} can directly be found from Eq. 21 for any N. For example in simplest case when N=2 the matrix L_{km} has the form $\begin{pmatrix} -2 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$.

Let us consider now this case more detail. Let $G_2\,(u,v,t) = au^2 + buv + cv^2$ is generating function of the model. Then Eq. 21 implies the next system of equations for evolution of coefficients a, b, c.

$$\frac{da}{dt} = -2a, \frac{db}{dt} = 2a - 2b, \frac{dc}{dt} = 2b. \tag{22}$$

Together with normalization condition $a+b+c=1$ system Eq. 22 allows one to give statistical description of the model at any time. In particular Eq. 22 implies that when t tends to infinity $\overline{n_1}$ tends to zero and $\overline{n_2}$ tends to 2.This completely agrees with solutions of dynamical equations in this case. From the other hand let us consider now the equation for generating function of this model obtained by Nicolis and Prigogine (see eq .10.66 in their book [6]) which in our notation has the form

$$\frac{dG}{dt} = (v-u)v\frac{\partial^2 G}{\partial u\partial v}. \tag{23}$$

In the case $N=2$ Eq. 23 implies the system of equations for coefficients of expansion $G_2 = au^2 + buv + cv^2$ which differs from 22 namely

$$\frac{da}{dt} = 0, \frac{db}{dt} = -b, \frac{dc}{dt} = b. \tag{24}$$

Eq. (24) imply that when t tends to infinity both quantities $\overline{n_1}, \overline{n_2}$ tend to nonzero values depending on initial conditions what obviously does not agree with dynamical equations. But when $N \gg 1$ it is easy to see that asymptotic behaviour solutions obtained from equations Eq. (21) and Eq. (23) are virtually identical. Thus in this example we see the benefits of approach proposed over the standard methods which do not allow to consider explicitly the discreteness of variables of the problem.

Finally in the last part of the paper using one concrete model of the population dynamics we want to demonstrate that approach proposed let one to give statistical description of the systems for which dynamical description in the framework of mean field theory looks as oversimplified. For this purpose we consider two "competing" kins of cannibals eating each other so that voracity of individuals in both kins is assumed to be distinctive. As one knows cannibalism is widespread in living nature and plays important role in evolution processes [7]. Besides many species possess special mechanisms which let them to recognize relatives and to avoid of their eating [8]. Let us assume that mutual eating is the major factor of changes of the number of individuals in both kins. Then evolution of the number of individuals $n_1$and n_2 in such model can be represented of simple system of equation of the form

$$\begin{aligned} \frac{dn_1}{dt} &= an_1n_2 - bn_1n_2, \\ \frac{dn_2}{dt} &= -an_1n_2 + bn_1n_2. \end{aligned} \tag{25}$$

It is easy to see that total number of individuals in this model N= $n_1 + n_2$ conserves. But dynamical description of the system with the help of Eq. (25) seems to be oversimplified. In particular it implies that for any N when $a > b$ and t tends to infinity $n_1$tends to N , and n_2 tends to zero. Now following the spirit of our method we will describe this system by the help of two operators $\widehat{R_1} = \lambda_1 \widehat{a_1} \widehat{a_2^+}$ and $\widehat{R_2} = \lambda_2 \widehat{a_1^+} \widehat{a_2}$. Show that such statistical version is completely consistent. Actually acting as in previous examples we can write the master equation for the density matrix of the system as

$$\frac{d\widehat{\rho}}{dt} = \left[\widehat{R_1}\widehat{\rho}, \widehat{R_1^+}\right] + \left[\widehat{R_2}\widehat{\rho}, \widehat{R_2^+}\right] + h.c.. \tag{26}$$

If again we are interesting in by solutions of the Eq. (26) of the form $\widehat{\rho} = \sum\limits_{n_1 n_2} |n_1 n_2\rangle \, \rho_{n_n n_2} \langle n_1 n_2|$ then for the generating function of the problem $G(u,v,t) = \sum\limits_{n_1 n_2} \rho_{n_1 n_2} u^{n_1} v^{n_2}$ we obtain the equation

$$\frac{dG}{dt} = a\,(u - v)\,\frac{d^2}{dudv}\,(uG) + b\,(v - u)\,\frac{d^2}{dudv}\,(vG)\,, \tag{27}$$

where $a = 2\lambda_2^2$ and $b = 2\lambda_1^2$. The equation Eq. (27) implies that generating function of stationary state of this model in the case when total number of individuals is equal to N can be represented in the form

$$G_{st}\,(u,v) = C_N \left[\frac{(bv)^{N+1} - (au)^{N+1}}{bv - au}\right] = C_N \left[(bv)^N + (bv)^{N-1}\,(au) + ...\right] \tag{28}$$

where C_N is the normalization factor, $C_N = \left(b^N + ab^{N-1} + ...\right)^{-1}$ Having in hands expression Eq. (28) for the generation function one can find all statistical characteristics of the model for any N. In particular for N=2 for the average values of individuals in both kins we obtain $\overline{n_1} = \frac{2a^2+ab}{b^2+ab+a^2}$, and $\overline{n_2} = \frac{2b^2+ab}{b^2+ab+a^2}$. Now we want to show that when N tends to infinity the properties of our statistical model will be similar to ones of the dynamical model Eq. (25).Let us assume that $b > a$ and let $\varkappa = \frac{a}{b}, (\varkappa < 1)$. Let us calculate now the ratio $\frac{\overline{n_1}}{\overline{n_2}}$ in this model. Using Eq. (28) we obtain

$$\frac{\overline{n_1}}{\overline{n_2}} = \frac{Na^N + (N-1)\,a^{N-1}b + ...}{Nb^N + (N-1)\,b^{N-1}a + ...} = \frac{N\varkappa^N + (N-1)\,\varkappa^{N-1} + ...\varkappa}{N \ + (N-1)\,\varkappa + ...\varkappa^{N-1}} \tag{29}$$

. Expression Eq. (29) can be represented in the next convenient form

$$\frac{\overline{n_1}}{\overline{n_2}} = \frac{\varkappa + \varkappa^2 \frac{\partial Lnf_N(\varkappa)}{\partial\varkappa}}{N - \varkappa\frac{\partial Lnf_N(\varkappa)}{\partial\varkappa}}, \tag{30}$$

where $f_N(\varkappa) = \frac{1-\varkappa^N}{1-\varkappa}$. It is easy to see that $\frac{\partial(Lnf_N)}{\partial\varkappa} = \frac{(N-1)\varkappa^N - N\varkappa^{N-1}+1}{(1-\varkappa)(1-\varkappa^N)}$. Obviously this expression tends to $\frac{1}{1-\varkappa}$ when $\varkappa < 1$ and N tends to infinity .Thus Eq. (30) implies that ratio $\frac{\overline{n_1}}{\overline{n_2}}$ tends to zero but this case is realized only for infinite population. From the other hand for any finite population of cannibals all its statistical properties can be well described by the approach proposed in our paper. In conclusion let us briefly summing up our consideration. We propose the statistical method of describing of various systems in physics,chemistry,ecology whose states can be represented by integers. Although the method proposed is based on quantum Lindblad equation nevertheless we have shown that it can be successfully used also for description of classical open systems with integer variables (at least in the cases when "Hamiltonian" of the open system is equal to zero) .All examples considered in present paper demonstrate that approach proposed results in consistent conclusions and allows one in several cases to eliminate essential inaccuracies of preceding considerations.

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